Chapter 2

Smith Chart

2.1 Definitions

In the loaded transmission line shown in Fig.2.1, the input impedance at a given position along the line is expressed as:

$$Z_{in}(\ell) = Z_c \frac{1+\Gamma(\ell)}{1-\Gamma(\ell)}. \tag{2.1}$$

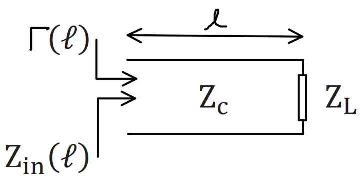

Figure 2.1: Loaded transmission line.

To simplify the analysis, we introduce normalized impedance values:

$$\bar{Z}_{in}(\ell) = \frac{Z_{in}(\ell)}{Z_c} \quad , \quad \bar{Z}_c = \frac{Z_c}{Z_c} = 1 \quad , \quad \bar{Z}_L = \frac{Z_L}{Z_c}. \tag{2.2}$$

This allows us to express the normalized input impedance as

$$\boxed{\bar{Z}_{in}(\ell) = \frac{R_{in}}{Z_c} + j\frac{X_{in}}{Z_c} = \bar{Z}_{in}}, \tag{2.3}$$

which leads to a more compact form

$$\boxed{\bar{Z}_{in} = \bar{R} + j\bar{X}}. \tag{2.4}$$

Applying equation (2.1) in terms of the normalized impedance

$$\boxed{\bar{Z}_{in} = \bar{R} + j\bar{X} = \frac{1+\Gamma(\ell)}{1-\Gamma(\ell)}}. \tag{2.5}$$

From this, we can derive the reflection coefficient as:

$$\boxed{\Gamma(\ell) = \frac{\bar{Z}_{in} - 1}{\bar{Z}_{in} + 1}}, \tag{2.6}$$

This transformation is known as the "bilinear transformation", which plays a crucial role in impedance matching and network analysis in RF circuit design.

2.2 Bilinear Transformation

Bilinear transformation, in general, is defined by the following coupled equations:

$$W = \frac{Z - 1}{Z + 1}, \tag{2.7}$$

$$Z = \frac{1 + W}{1 - W} \tag{2.8}$$

where $Z = x + jy$, and $W = u + jv$.

From equation (2.4), the normalized input impedance is represented as

$$\bar{Z}_{in} = \bar{R} + j\bar{X}. \tag{2.9}$$

By assuming:

$$\Gamma(\ell) = u + jv, \tag{2.10}$$

we can substitute Z and W in equations (2.7) and (2.8), respectively.

Inserting equations (2.9) and (2.10) into equation (2.8), we obtain:

$$\bar{R} + j\bar{X} = \frac{1 + u + jv}{1 - u - jv} = \frac{(1 + u + jv)(1 - u + jv)}{(1 - u)^2 + v^2}. \tag{2.11}$$

Rearranging, we derive:

$$\bar{R} + j\bar{X} = \frac{1 - u^2 - v^2}{(1 - u)^2 + v^2} + j\frac{2v}{(1 - u)^2 + v^2}. \tag{2.12}$$

From this, we identify:

$$\bar{R} = \frac{1 - u^2 - v^2}{(1 - u)^2 + v^2}, \tag{2.13}$$

$$\bar{X} = \frac{2v}{(1 - u)^2 + v^2}. \tag{2.14}$$

Rewriting equation (2.13), we obtain:

$$(u - 1)^2 + v^2 - \frac{2v}{\bar{X}} = 0, \tag{2.15}$$

or equivalently:

$$(u-1)^2 + \left(v - \frac{1}{\bar{X}}\right)^2 = \frac{1}{\bar{X}}, \tag{2.16}$$

which represents a circle in the uv-plane, commonly referred to as the "constant reactance circle."

Similarly, equation (2.15) can be rearranged into:

$$\left(u - \frac{\bar{R}}{\bar{R}+1}\right)^2 + v^2 = \frac{1}{(\bar{R}+1)^2}. \tag{2.17}$$

This equation also represents a circle in the uv-plane, known as the "constant resistance circle."

2.3 Constant Resistance Circles

"Constant resistance circles" are obtained by plotting (2.17) on the uv-plane for different values of $\bar{R}$. Some of the constant resistance circles are shown in Fig.2.2(a).

2.4 Constant Reactance Circles

"Constant reactance circles" are obtained by plotting (2.16) on the uv-plane for different values of $\bar{X}$. Some of the constant reactance circles are shown in Fig.2.2(b).

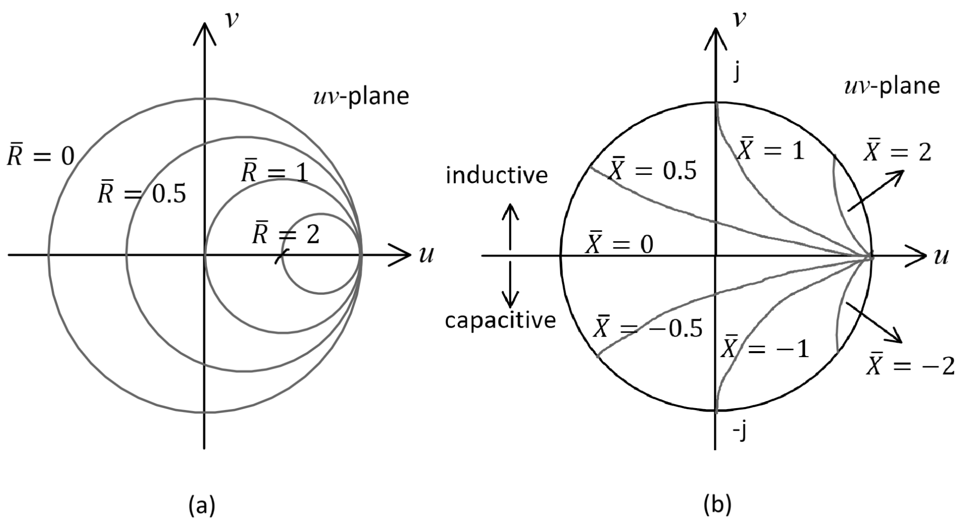

Figure 2.2: (a) Constant resistance circles, (b) Constant reactance circles.

From (2.10), $\Gamma(\ell) = u + jv = \Gamma_L e^{-2j\beta\ell}$ for lossless lines, and since $|\Gamma_L| \leq 1$, we are confined in a unit circle. Combining the constant $\bar{R}$ and $\bar{X}$

circles in this unit circle, we obtain what is known as the "Smith Chart", as shown in Fig.2.3.

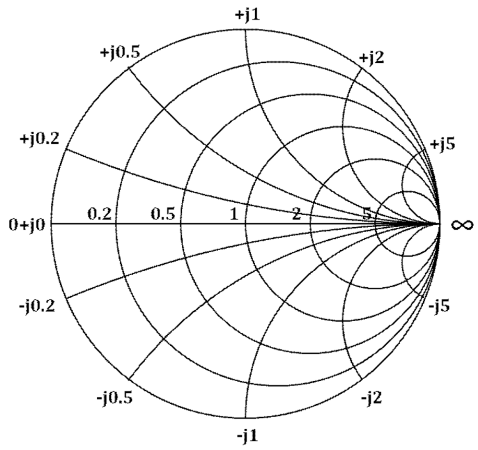

Figure 2.3: The Smith chart.

2.5 Mapping

Considering the points A, B, C, D, E and P on the $\bar{R} + j\bar{X}$ plane, we want to map these points on the Smith chart on the uv-plane. The results are shown in Fig.2.4.

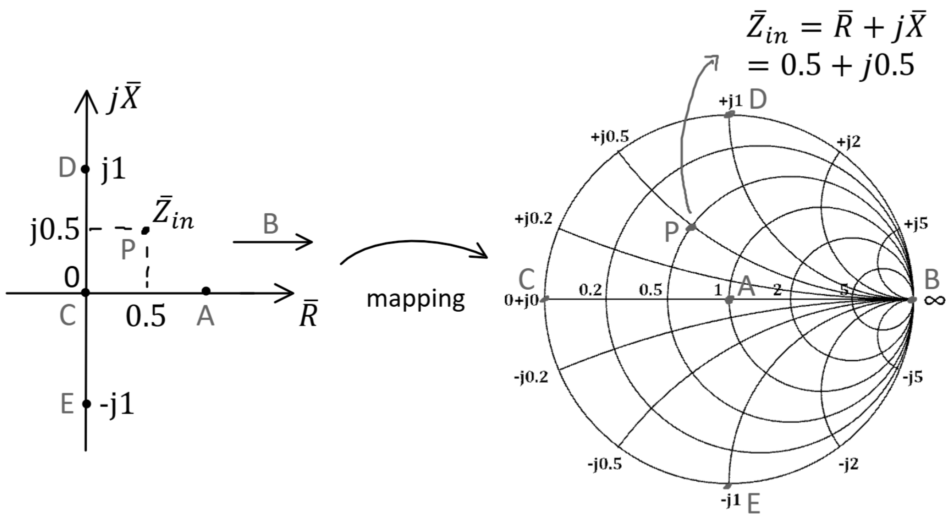

Figure 2.4: Mapping on the Smith chart.

2.6 VSWR Circles

For lossless lines (or low-loss lines), the VSWR circle is definable. For a transmission line shown in Fig.2.5, the reflection coefficient is given by

$$\Gamma(\ell) = \rho e^{j(\theta - 2\beta\ell)} \tag{2.18}$$

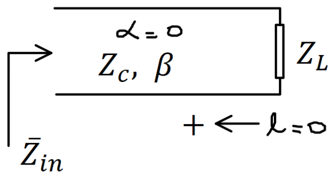

Figure 2.5: Lossless transmission line.

Thus, moving along the constant VSWR circle, ρ remains constant, but $\bar{Z}_{in}$ and $(\theta - 2\beta\ell)$ changes.

As an example, let $\bar{Z}_L = 0.2 + j0.5$ (Ω). First, this point is marked on the Smith chart, and the constant VSWR circle is drawn with respect to the center of the chart as shown in Fig.2.6. Moving along the constant VSWR circle in the clockwise direction is equivalent to moving towards the generator inside the transmission line.

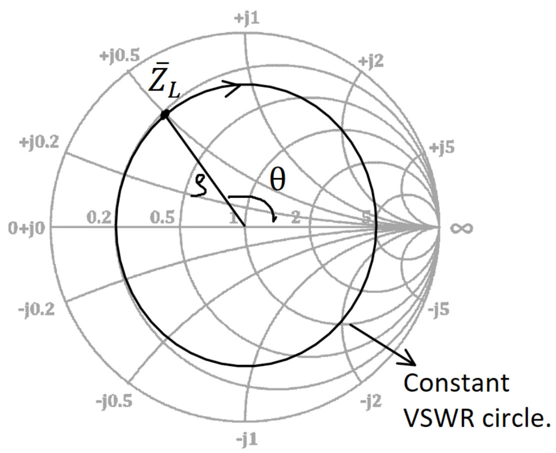

Figure 2.6: The constant VSWR circle.

It should be noted that across the load, at $\ell = 0$, $\Gamma(0) = \Gamma_L = \rho e^{j\theta}$, and since $\Gamma(\ell) = \rho e^{j(\theta - 2\beta\ell)}$, as ℓ increases, $(\theta - 2\beta\ell)$ decreases.

2.7 Reading of VSWR on the Smith Chart

When the constant VSWR circle intersects the real axis, where $\bar{R} > 1$, S (VSWR) is the value of $\bar{R}$ at that point.

* * *

Proof:
On the real axis, $Z_{in} = \bar{R} > 1$, $\bar{X} = 0$, and the VSWR is calculated as:

$$S = \frac{1+\rho}{1-\rho} = \frac{1+|\Gamma(\ell)|}{1-|\Gamma(\ell)|} = \frac{1+\frac{\bar{R}-1}{\bar{R}+1}}{1-\frac{\bar{R}-1}{\bar{R}+1}} > 0. \tag{2.19}$$

Since $\bar{R} > 1$, the previous expression becomes:

$$\boxed{S = \frac{\bar{R}+1+\bar{R}-1}{\bar{R}+1-\bar{R}+1} = \bar{R}}. \tag{2.20}$$

2.8 VSWR for Lossy Lines

For lossy transmission lines, the amplitude of the reflected voltage wave diminishes as it propagates along the line. Consequently, the concept of Voltage Standing Wave Ratio (VSWR) becomes less meaningful, as the standing wave pattern is no longer well-defined. On the Smith chart, the VSWR decreases as one moves toward the generator, as illustrated in Fig.2.7.

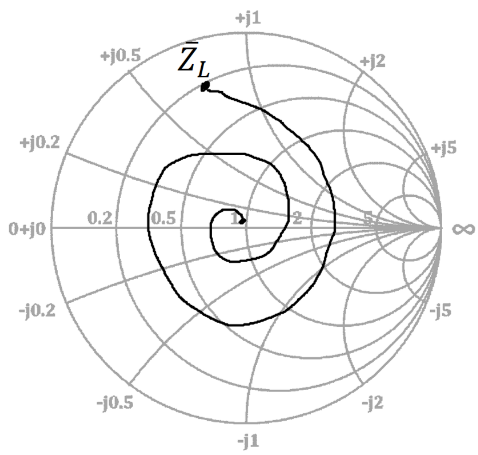

Figure 2.7: VSWR curve for lossy lines.

2.9 Periphery of the Smith Chart

Consider the transmission line shown in Fig.2.8.

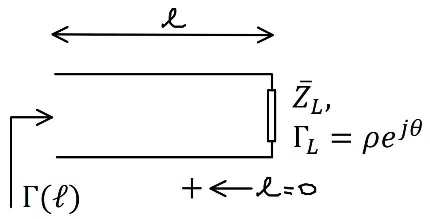

Figure 2.8: Loaded transmission line.

The voltage reflection coefficient is given by:

$$\Gamma(\ell) = \rho e^{-2\alpha\ell} e^{j(\theta - 2\beta\ell)}, \tag{2.21}$$

where:

$$2\beta\ell = 2\frac{\omega}{v_p}\ell = 2\frac{2\pi f}{\lambda f}\ell. \tag{2.22}$$

Rearranging,

$$2\beta\ell = 4\pi\frac{\ell}{\lambda} \quad \text{(radians)}, \tag{2.23}$$

where $\boxed{\frac{\ell}{\lambda}}$ is referred to as the "normalized length", or "electrical length".

Equation (2.23) can be converted to degrees as

$$2\beta\ell = 4\pi\frac{\ell}{\lambda}\frac{180}{\pi} = \frac{4\ell}{\lambda}180 \quad \text{(degrees)}. \tag{2.24}$$

Thus, when ℓ varies from 0 to 0.5, $2\beta\ell$ varies from 0° to 360°. This means that in a lossless transmission line, at every $\lambda/2$ interval, $\Gamma(\ell)$ and $\bar{Z}$ repeat themselves, completing a full rotation around the Smith chart. This periodic behavior is illustrated in Fig.2.9.

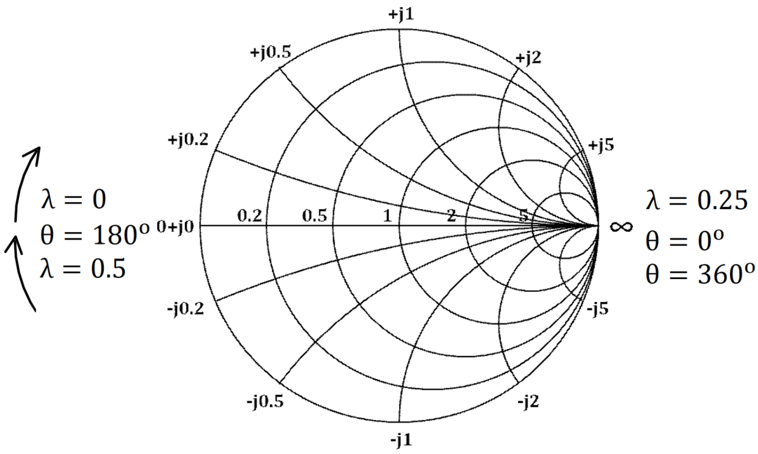

Figure 2.9: Periphery of the Smith chart.

Additionally, the Smith chart can be used for admittance values. In other words, it can be converted from an impedance chart to an admittance chart and vice versa.

2.9.1 Example

A 50 (Ω) air-filled transmission line is terminated with a 100 (Ω) load resistance. Find the load reflection coefficient and the input impedance at $\ell = 50$ (cm) away from the load using the Smith chart. Given $f = 240$ (MHz).

Solution

The problem description is shown in Fig.2.10.

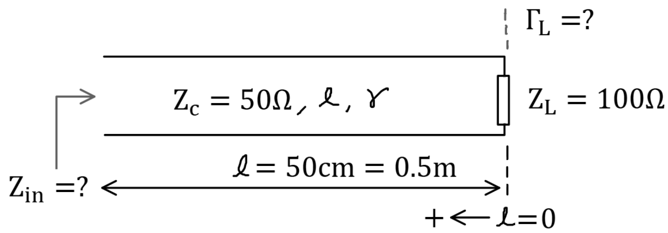

Figure 2.10: Example problem schematics.

In the first step, the load impedance, Z_L is normalized:

$$\bar{Z}_L = \frac{100}{50} = 2\ (\Omega). \tag{2.25}$$

Then, this value is marked on the Smith chart, and the constant VSWR circle is drawn, as shown in Fig.2.11.

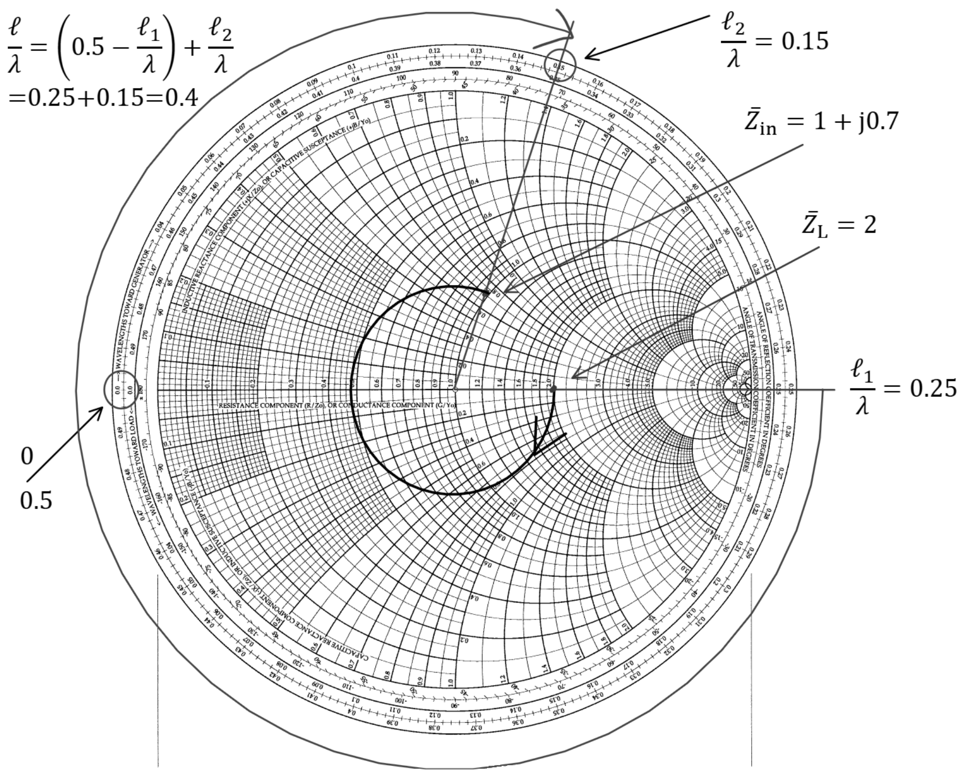

Figure 2.11: Smith chart solution for Z_{in}.

Next, this point is rotated clockwise along the constant VSWR circle until a distance of 50 (cm) is traced on the periphery of the Smith chart. Since the periphery is scaled for normalized lengths, 50 (cm) is normalized as:

$$\frac{\ell}{\lambda} = \frac{0.5}{5/4} = \frac{2}{5} = 0.4, \tag{2.26}$$

where $\lambda = c/f = (3 \times 10^8)/(240 \times 10^6) = 5/4$ (m)=125 (cm).

To rotate by $\ell/\lambda = 0.4$, first, a half of the Smith chart, $\ell/\lambda = 0.25$, is rotated, and then another $\ell/\lambda = 0.15$ distance is rotated as shown in Fig.2.11. This is due to the fact that the Smith chart resets at the leftmost point where $\ell/\lambda = 0$.

At this final location on the VSWR circle, the impedance value on the Smith chart represents the normalized input impedance observed at 50 (cm) away from the load. For this example, this value is observed to be $\bar{Z}_{in} = 1 + j0.7$ (Ω). De-normalizing this value gives

$$Z_{in} = \bar{Z}_{in} Z_c = (1 + j0.7) \times 50 = 50 + j37 \quad (\Omega) \tag{2.27}$$

which is the same result as in the analytic solution.

To find Γ_L, a vertical line is drawn from the intersection of the constant VSWR circle with the real axis on the left of the chart as shown in Fig.2.12. This line crosses the voltage reflection coefficient scale located at the bottom of the Smith chart. The intersection of the line on this scale gives the value for the magnitude of the load reflection coefficient, $|\Gamma_L| = \rho$.

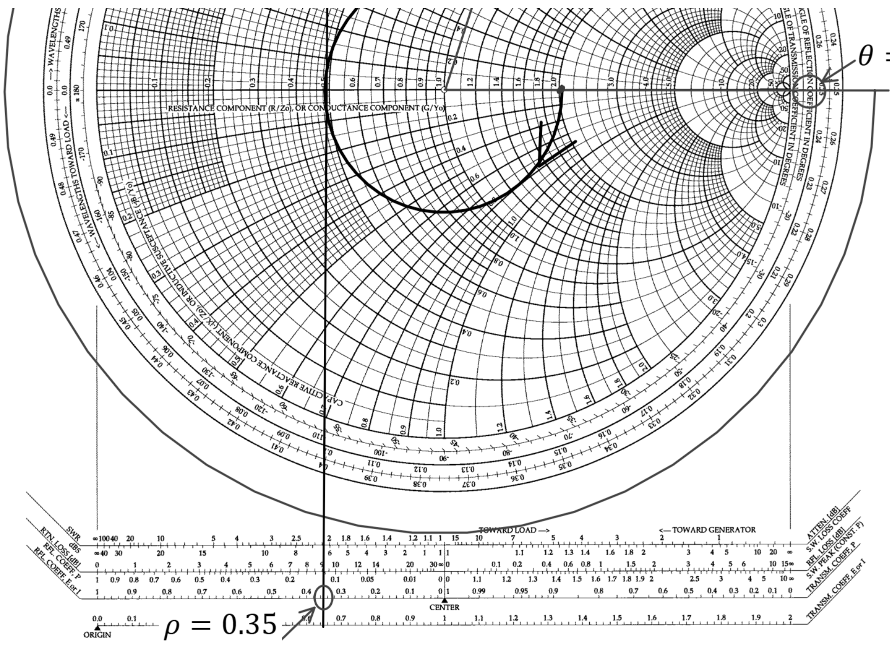

Figure 2.12: Smith chart solution for $\Gamma_L = \rho e^{j\theta}$.

Finally, it is worth noting that some differences between the analytic and the Smith chart solutions are acceptable due to inaccuracies in reading chart values.

ADS Simulations

A frequency-domain simulation (S-parameter simulation) is conducted in ADS to generate the constant VSWR circle. The parameter θ is varied from 72° to 360°, corresponding to the points $\bar{Z}_{in} = 1 + j0.7$ (Ω) and $\bar{Z}_L = 2$ (Ω), on the Smith chart. The circuit schematics used for this simulation are shown in Fig. 2.13.

To plot the VSWR circle, the locations of the input impedance Z_{in} and the load impedance Z_L are identified by plotting S_{11} and S_{22} on the Smith chart. The reflection coefficient $\Gamma(\ell)$ is then defined and used to determine the corresponding impedance values $\bar{Z}$ with respect to a reference impedance of 50 (Ω). The resulting graph is presented in 2.14.

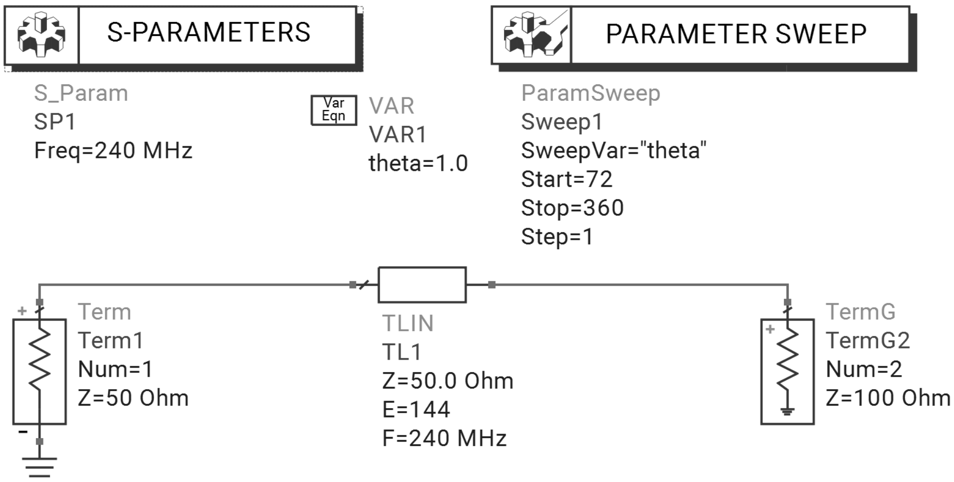

Figure 2.13: ADS schematics for drawing constant VSWR circle.

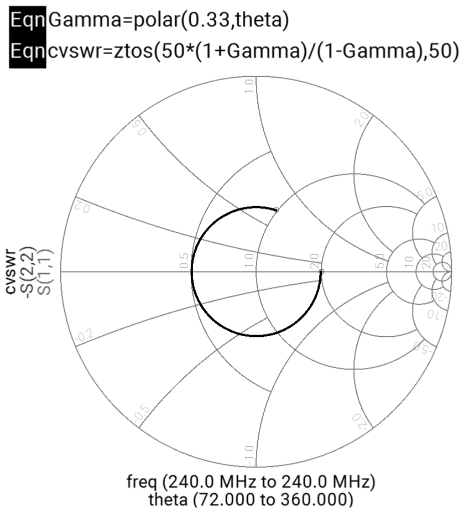

Figure 2.14: ADS simulation results: constant VSWR circle.

2.10 Admittance

The normalized impedance is given by:

$$\bar{Z} = \frac{1+\Gamma(\ell)}{1-\Gamma(\ell)}, \tag{2.28}$$

and the normalized admittance is defined as:

$$\bar{Y} = \frac{1}{\bar{Z}} = \frac{1-\Gamma(\ell)}{1+\Gamma(\ell)} \tag{2.29}$$

This reveals that the reflection coefficient undergoes a sign reversal. In practical terms, switching between impedance and admittance values involves multiplying the reflection coefficient by -1, which corresponds to a $\theta = 180^{\circ}$ rotation on the Smith chart ($e^{j\pi} = -1$).

Thus, when applying a 180° symmetry to $\bar{Z}$ on the Smith chart, we obtain $\frac{1}{\bar{Z}} = \bar{Y}$, representing the admittance. This relationship is visually depicted in Fig.2.15.

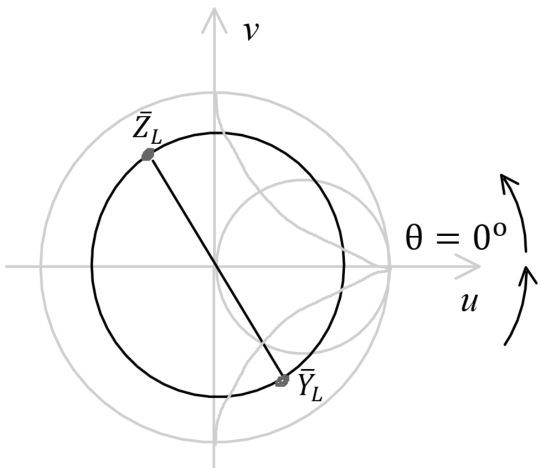

Figure 2.15: Admittance on the Smith chart.

2.11 Stubs

Stubs are essential components in transmission line engineering, widely used for impedance matching and reactive tuning. They can be categorized as

- Short-circuited stub – This consists of a transmission line terminated with a short circuit.
- Open-circuited stub – This is a transmission line with an open-circuit termination.
- Application in impedance matching – Reactive elements, such as stubs, are commonly employed to adjust impedance and optimize signal transmission.

2.12 Short-Circuited Transmission Line (Lossless)

Consider a short-circuited transmission line, as illustrated in Fig.2.16.

The input impedance of the transmission line is given by:

$$Z_{in} = Z_c \frac{Z_L + jZ_c \tan \beta\ell}{Z_c + jZ_L \tan \beta\ell}. \tag{2.30}$$

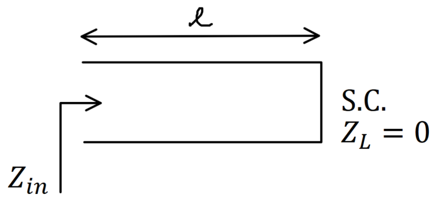

Figure 2.16: Short-circuited transmission line.

By substituting $Z_L = 0$, we obtain

$$\boxed{Z_{in} = jZ_c \tan \beta\ell}. \tag{2.31}$$

This result shows that Z_{in} is purely reactive. Depending on the electrical length ℓ, the impedance behaves like either an inductor or a capacitor. Specifically:

– For $0 < \ell < \lambda/4$, the impedance exhibits inductive characteristics.

– For $\lambda/4 < \ell < \lambda/2$, the impedance behaves capacitively.

Since the phase constant is $\beta = \frac{2\pi}{\lambda}$, we can express the electrical length as $\beta\ell = \frac{2\pi\ell}{\lambda}$. This leads to the following observations:

– $\tan \beta\ell > 0$ for $0 < \ell < \lambda/4$, indicating inductive behavior.

– $\tan \beta\ell < 0$ for $\lambda/4 < \ell < \lambda/2$, signifying capacitive characteristics.

These impedance variations are illustrated in Fig.2.17.

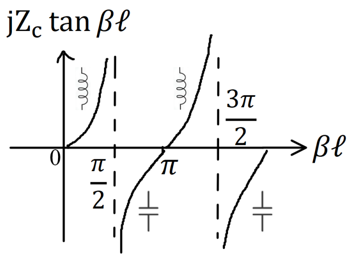

Figure 2.17: Z_{in} for short circuited transmission line.

To determine the equivalent inductance and capacitance: For $0 < \ell < \lambda/4$:

$$\omega L_{eq} = Z_c \tan \beta\ell \Rightarrow L_{eq} \text{ is evaluated.} \tag{2.32}$$

For $\lambda/4 < \lambda < \lambda/2$:

$$\frac{1}{\omega C_{eq}} = Z_c \tan \beta\ell \Rightarrow C_{eq} \text{ is evaluated.} \tag{2.33}$$

2.13 Open-Circuited Transmission Line (Lossless)

In the case of an open-circuited transmission line as shown in Fig.2.18, the input impedance is derived from the general transmission line equation:

$$Z_{in} = Z_c \frac{Z_L + jZ_c \tan \beta\ell}{Z_c + jZ_L \tan \beta\ell}. \tag{2.34}$$

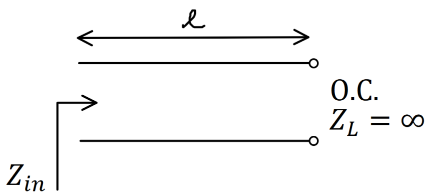

Figure 2.18: Open-circuited transmission line.

By substituting $Z_L = \infty$, the expression simplifies to:

$$Z_{in} = \frac{Z_c}{j \tan \beta\ell} = -jZ_c \cot \beta\ell. \tag{2.35}$$

Thus, the input admittance is given by:

$$\frac{1}{Z_{in}} = \boxed{Y_{in} = jY_c \tan \beta\ell}, \tag{2.36}$$

where the characteristic admittance is defined as:

$$\boxed{Y_c = \frac{1}{Z_c}} \quad \text{(Siemens)} \tag{2.37}$$

is the "characteristic admittance".

The input impedance exhibits capacitive behavior for $0 < \ell < \lambda/4$, meaning it acts as a capacitive reactance in this range. Conversely, for $\lambda/4 < \ell < \lambda/2$, the impedance becomes inductive. The variation of Z_{in} as a function of line length ℓ is illustrated in Fig. Fig.2.19.

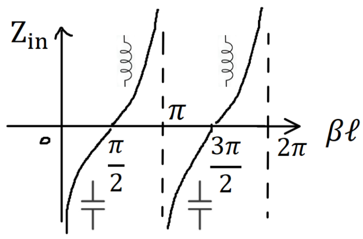

Figure 2.19: Z_{in} for open circuited transmission line.

2.14 Problems

1. A 50 (Ω) air-filled transmission line operates without loss and carries a 300 (MHz) sinusoidal signal. The line is terminated with a load impedance of $20 + j30$ (Ω). Using the Smith chart, determine
a. the reflection coefficient at the load.
b. the input impedance at a distance of $\ell = 20$ (cm) from the load.

2. Using the Smith chart, calculate the input impedance of a short-circuited, air-filled stub with a length of 6 (cm) at a frequency of 1 (GHz).

Chapter 3

Impedance Matching

Impedance matching refers to adjusting a load impedance Z_L to meet the condition:

$$\boxed{Z_L = Z_S^*}, \tag{3.1}$$

where Z_S is the source impedance, connected to Z_L.

However, in microwave and RF applications, we are typically matching a complex load $Z_L = R_L + jX_L$ to a real characteristic impedance Z_c of a transmission line. Since transmission lines can be modeled as having a generator with an output impedance equal to Z_c, the corresponding matching topology is illustrated in Fig.3.1.

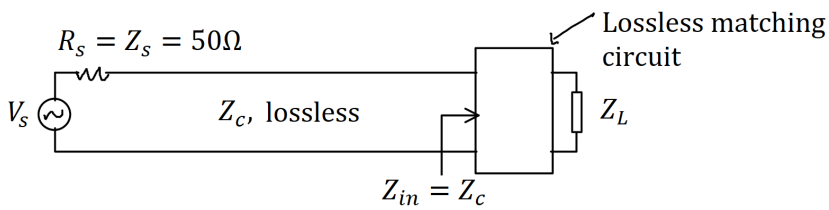

Figure 3.1: Impedance matching in transmission lines.

Thus, Impedance matching ensures maximum power transfer from the transmission line to the load without any reflections. A general topology for impedance matching in transmission lines is shown in Fig.3.2.

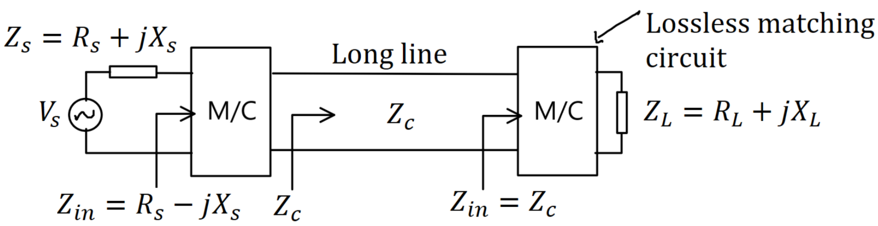

Figure 3.2: Impedance matching in transmission lines.

3.1 Quarter-wave and Half-wave Transformers

Consider the transmission line configuration shown in Fig.3.3.

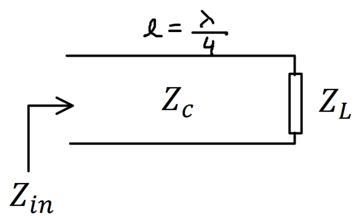

Figure 3.3: Quarter-wave line.

When the line length is $\ell = \lambda/4$, we use the general impedance transformation equation:

$$Z_{in} = Z_c \frac{Z_L + jZ_c \tan \beta\ell}{Z_c + jZ_L \tan \beta\ell}. \tag{3.2}$$

Substituting $\beta\ell = \pi/2$, i.e., a quarter wavelength, yields:

$$Z_{in} = Z_c \frac{jZ_c}{jZ_L} = \frac{Z_c^2}{Z_L}, \tag{3.3}$$

or equivalently,

$$\boxed{Z_{in} Z_L = Z_c^2}. \tag{3.4}$$

This relationship defines the quarter-wave transformer, which enables impedance matching between the load and the input by appropriately selecting the characteristic impedance Z_c of the transformer section.

When $\ell = \lambda/2$, we have $\tan \beta\ell = \tan \pi = 0$, and the equation reduces to $\boxed{Z_{in} = Z_L}$. This configuration behaves as a 1:1 transformer with phase reversal, meaning the input and load impedances are identical in magnitude, but the signal experiences a 180° phase shift.

3.2 Matching Techniques Involving Quarter-wave Transformers

In the following sections, we examine impedance matching techniques based on the nature of the load impedance Z_L.

3.2.1 Matching a Purely Real Load Impedance

Consider the quarter-wave transformer with characteristic impedance $Z_{c\lambda}$, as illustrated in Fig.3.4. When the load is purely resistive, i.e., $Z_L = R_L$,

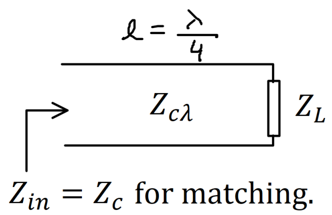

Figure 3.4: Quarter-wave transformer.

the input impedance becomes:

$$Z_{in} = \frac{Z_{c\lambda}^2}{Z_L}. \tag{3.5}$$

To achieve a matched condition ($Z_{\text{in}} = Z_0$), we select the transformer's characteristic impedance as:

$$\boxed{Z_{c\lambda} = \sqrt{Z_{in} Z_L}}. \tag{3.6}$$

This approach ensures maximum power transfer between the source and load by transforming R_L into the system's characteristic impedance via the quarter-wave section.

3.2.2 Example

Match 200 (Ω) load resistance to a 50 (Ω) transmission line.

Solution

Since Z_L is real, we can use the quarter-wave transformer with a characteristic impedance

$$Z_{c\lambda} = \sqrt{Z_{in} Z_L} = \sqrt{50 \times 200} = 100\ (\Omega). \tag{3.7}$$

Thus, the following transmission line in Fig.3.5 establishes matching.

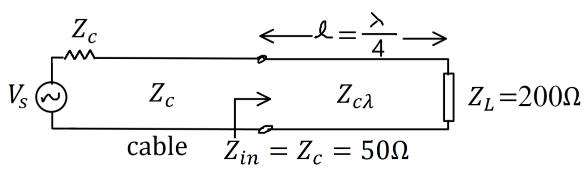

Figure 3.5: Solution of the matching problem.

ADS Simulations

In ADS, the line length is maintained at $\lambda/4$, and the characteristic impedance, $Z_{c\lambda} \in [50, 150]$ (Ω) is varied, as shown in Fig.3.6. The results in Fig.3.7 show that at the center of the Smith chart, where matching occurs, $Z_{c\lambda} = 100$ (Ω) as expected.

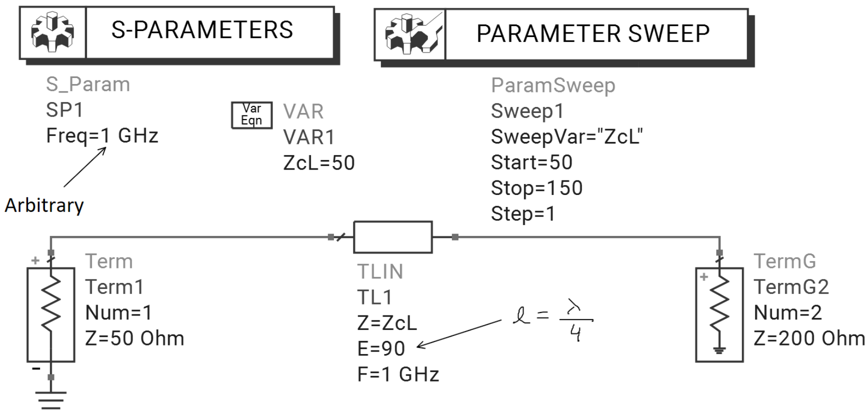

Figure 3.6: ADS schematics for real impedance matching problem.

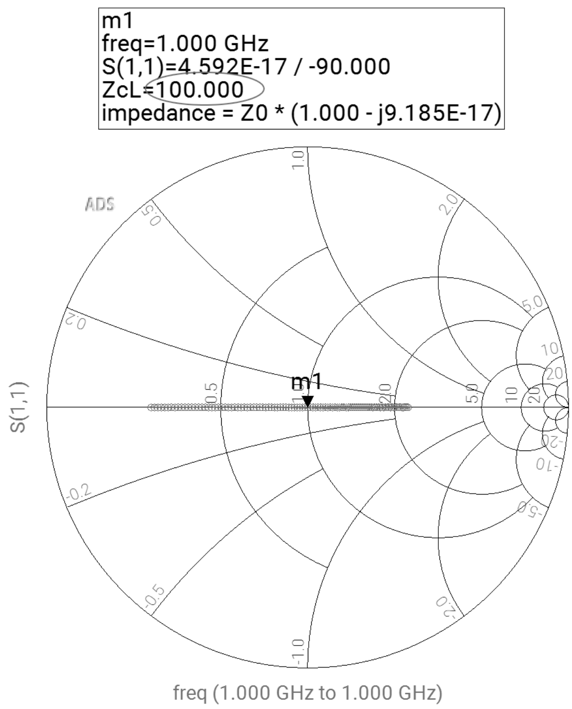

Figure 3.7: ADS output for real impedance matching problem.

3.2.3 Matching a Complex Load Impedance Using a Transmission Line and a Quarter-wave Transformer

If Z_L is complex, i.e. $Z_L = R_L + jX_L$, then we can use the following technique: Use a transmission line section of length ℓ, such that Z'_{in} is purely real as shown in Fig.3.8, then use a $\lambda/4$ transformer as before.

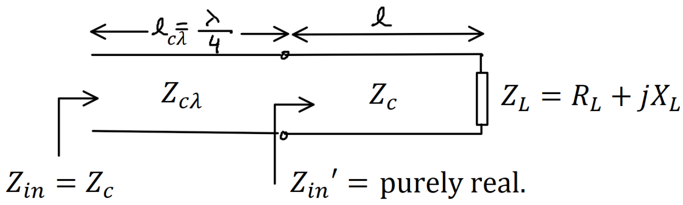

Figure 3.8: Quarter-wave transformer matching for a complex load.

3.2.4 Example

Given a load impedance of $Z_L = 150 - j100$ (Ω), match it to a transmission line with $Z_c = 50$ (Ω) at $f = 1$ (GHz) using a short transmission line followed by a quarter-wave transformer.

Solution

We begin by normalizing the load:

$$\bar{Z}_L = \frac{Z_L}{Z_c} = \frac{150 - j100}{50} = 3 - j2 \ (\Omega). \tag{3.8}$$

This point is marked on the Smith chart. Drawing a constant VSWR circle through this point, we move clockwise (toward the generator) along the circle until we reach the real axis — this corresponds to the point where the input impedance is purely real. These steps are shown in Fig.3.9.

Next, we mark this point on the Smith chart, and draw a constant VSWR circle from the center of the chart as shown in Fig.3.9. Moving towards the generator, we rotate clockwise around the constant VSWR circle until we intersect the real axis where Z_{in} will be purely real.

Using the outer scale of the Smith chart, we find the length of the transmission line section:

$$\frac{\ell}{\lambda} = 0.5 - 0.276 = 0.224. \tag{3.9}$$

Since $\lambda = c/f = 30$ (cm), this yields: $\ell = (0.224)(30) = 6.72$ (cm).

From the Smith chart, the normalized input impedance at this point is $\bar{Z}'_{in} = 0.22$, which de-normalizes to:

$$Z'_{in} = \bar{Z}'_{in} Z_c = 0.22 \times 50 = 11 \ (\Omega). \tag{3.10}$$

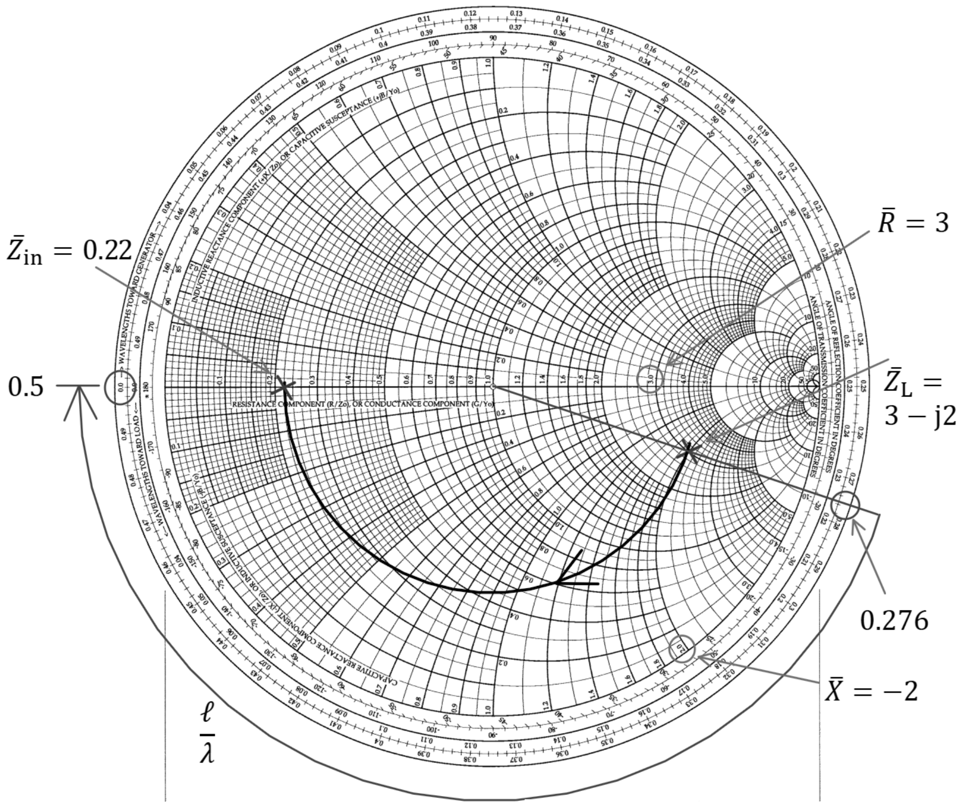

Figure 3.9: Smith chart solution for complex load matching problem.

To match this impedance to $Z_c = 50\ \Omega$, we use a quarter-wave transformer with:

$$Z_{c\lambda} = \sqrt{Z'_{in} Z_c} = \sqrt{11 \times 50} = 23.45\ (\Omega). \tag{3.11}$$

The corresponding length is: $\ell_{c\lambda} = \lambda/4 = 30/4 = 7.5$ (cm).

Thus, the final matching circuit consists of a 6.72 (cm) long transmission line followed by a 7.5 (cm) quarter-wave transformer with $Z_{c\lambda} = 23.45\ (\Omega)$, as shown in Fig.3.10.

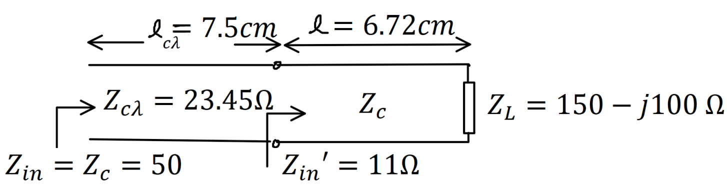

Figure 3.10: The solution of the matching problem with complex Z_L.

ADS Simulations

To observe the simulation results, the analytically determined values are inserted into the ADS schematic, as shown in Fig.3.11. The corresponding output is presented in Fig.3.12, showing an input impedance of approximately $Z_{in} \approx 50$ (Ω) and a reflection coefficient $\Gamma_{in} = S(1,1) \approx 0$.

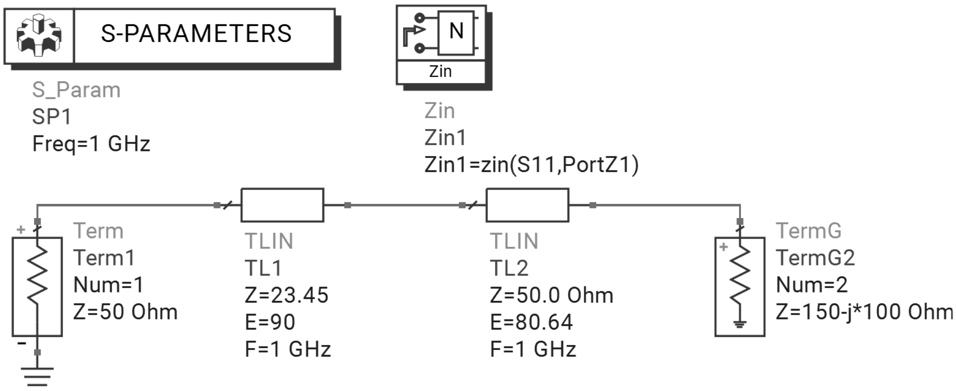

Figure 3.11: ADS schematics of the matching problem with complex Z_L.

freq	Zin1	mag(S(1,1))
1.000 GHz	48.842 + j0.512	0.013

Figure 3.12: ADS results for the matching problem with complex Z_L.

These results confirm successful impedance matching, with a voltage reflection coefficient magnitude of $\rho = |S(1,1)| = 0.013$. The slight deviation from perfect matching arises from rounding and numerical imprecision inherent in the analytical calculation.

3.2.5 Matching a Complex Load Impedance Using a Short-circuited Stub and a Quarter-wave Transformer

In this technique, we replace the transmission line segment of length ℓ used in the previous method with a stub. The stub serves the same function: it cancels out the reactive component of the input impedance, enabling the quarter-wave transformer to perform impedance matching to $Z_c = 50$ (Ω). This topology is illustrated in Fig.3.13.

The stub is connected in parallel, i.e., a shunt connection, with the load impedance, so that their admittances add to yield the input admittance Y'_{in}. We adjust the stub's length, ℓ_2, until the resulting admittance is purely real.

The load admittance is given by

$$Y_L = G_L + jB_L \quad (S), \tag{3.12}$$

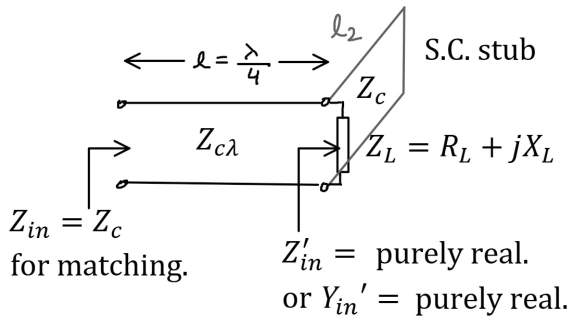

Figure 3.13: Alternative matching technique for complex Z_L.

where G_L is the conductance and B_L is the susceptance of the load.

The normalized admittance becomes

$$\bar{Y}_L = \frac{Y_L}{Y_c}, \tag{3.13}$$

with the characteristic admittance defined as

$$Y_c = \frac{1}{Z_c}. \tag{3.14}$$

The normalized total input admittance is the sum of the normalized load admittance and the stub's normalized susceptance:

$$\bar{Y}'_{in} = \bar{Y}_L + j\bar{B}_{sc}, \tag{3.15}$$

where $j\bar{B}_{sc}$ represents the reactive input admittance of the short-circuited stub at the load end.

To achieve a purely real input admittance, we require $\bar{Y}'_{in} = \bar{G}_L$. Substituting gives:

$$\bar{Y}'_{in} = \bar{G}_L + j\bar{B}_L + j\bar{B}_{sc} = \bar{G}_L \text{ (real)}. \tag{3.16}$$

which implies

$$j\bar{B}_L + j\bar{B}_{sc} = 0, \tag{3.17}$$

Thus,

$$\boxed{\bar{B}_{sc} = -\bar{B}_L}. \tag{3.18}$$

This equation determines the required stub length ℓ_2; we adjust ℓ_2 until (3.18) is satisfied. Finally, the actual load conductance is given by

$$G_L = Y_c\bar{G}_L \tag{3.19}$$

To determine the characteristic impedance for the quarter-wave transformer, we apply

$$Z_{c\lambda} = \sqrt{Z_c\frac{1}{G_L}}. \tag{3.20}$$

3.2.6 Example

Given $Z_L = 150 - j100$ (Ω), match this load to a $Z_c = 50$ (Ω) coaxial cable at $f = 1$ (GHz), using a stub of length ℓ_2, and a quarter-wave transformer (assume all air-filled transmission lines).

Solution

We start by normalizing the load impedance:

$$\bar{Z}_L = \frac{Z_L}{Z_c} = \frac{150 - j100}{50} = 3 - j2\ (\Omega). \tag{3.21}$$

Then, the corresponding normalized admittance is calculated as:

$$\bar{Y}_L = \frac{1}{\bar{Z}_L} = \frac{1}{3 - j2} = 0.23 + j0.15\ (\mathrm{S}). \tag{3.22}$$

Since changing ℓ_2 does not affect the conductance G_L, we move along the constant-conductance circle on the Smith chart. We continue this motion until the input admittance becomes: $\bar{Y}'_{in} = 0.23 + j0.15 + j\bar{B}_{sc}$. To achieve matching, the stub must provide a susceptance of $\bar{B}_{sc} = -0.15$.

To determine the stub length ℓ_2, we refer to the circuit in Fig.3.14. When using the Smith chart as an admittance chart, a short circuit corresponds to the rightmost point. From there, we rotate clockwise along the constant-conductance circle until we reach the susceptance value $\bar{B}_{sc} = -0.15$, as illustrated in Fig.3.15.

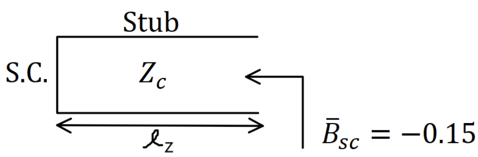

Figure 3.14: Finding the stub length, ℓ_2.

Thus, the normalized stub length is:

$$\frac{\ell_2}{\lambda} = 0.476 - 0.25 = 0.226. \tag{3.23}$$

From this, the physical length of the stub becomes:

$$\ell_2 = 0.226 \times 30cm = 6.78\ (\mathrm{cm}), \tag{3.24}$$

where the wavelength is taken as $\lambda = 30$ (cm).

To determine the normalized input impedance, we compute:

$$\bar{Z}'_{in} = \frac{1}{\bar{Y}'_{in}} = \frac{1}{0.23} = 4.35. \tag{3.25}$$

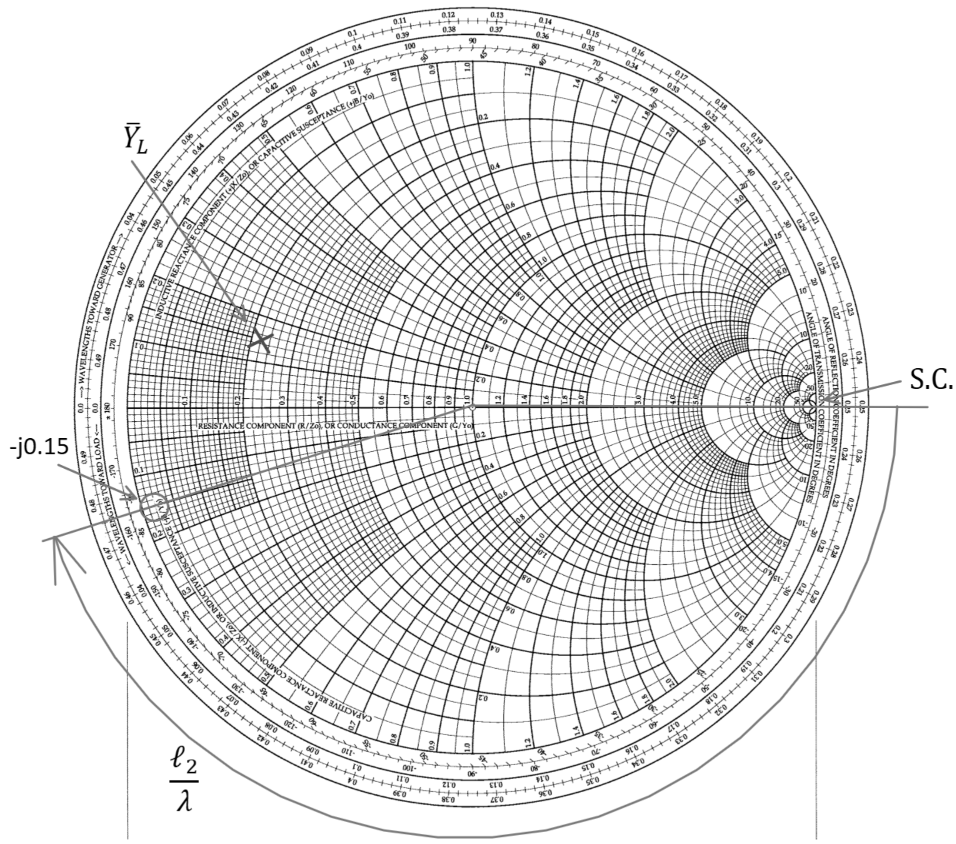

Figure 3.15: Matching a complex load using a stub and a quarter-wave transformer.

Thus, the actual input impedance becomes:

$$Z'_{in} = 4.35 \times Z_c = 4.35 \times 50 = 217.5 \quad (\Omega), \tag{3.26}$$

The characteristic impedance required for the quarter-wave transformer is then:

$$Z_{c\lambda} = \sqrt{Z'_{in} \times Z_c} = \sqrt{217.5 \times 50} = 104 \quad (\Omega). \tag{3.27}$$

ADS Simulations

The ADS circuit schematic is shown in Fig.3.16, utilizing the values derived from the analytical solution.

The stub length is converted into degrees as

$$\ell_2 = 6.78cm \times \frac{360^\circ}{\lambda} = 6.78cm \times \frac{360^\circ}{30cm} = 81.36^\circ. \tag{3.28}$$

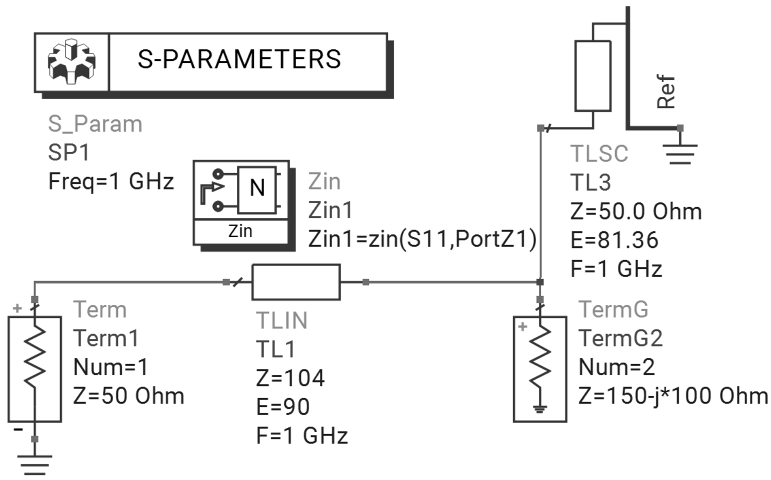

Figure 3.16: ADS circuit schematics for the short-circuited stub matching problem.

freq	Zin1	abs(S(1,1))
1.000 GHz	49.920 + j0.410	0.004

Figure 3.17: ADS results for the short-circuited stub matching problem.

The simulation results, displayed in Fig.3.17, confirm that impedance matching has been achieved, with a reflection coefficient magnitude of $\rho = 0.004$.

3.3 Matching with a Single Stub and a Variable Length Line, All Having the Same Zc

Matching sections involving transmission lines with all Z_c can be realized using a variable line of length ℓ_1 and a single stub of length ℓ_2 as shown in Fig.3.18.

In this technique, we find the normalize complex load admittance $\bar{Y}_L$ on the Smith chart, and move toward the generator (clockwise direction) on the constant VSWR circle until we intersect the unity conductance circle ($\bar{Y} = 1$). At this point, the input impedance becomes $\bar{Y}_{in} = 1 + j\bar{B}_{in}$. Finally, we use a short-circuited stub to cancel the susceptance term so that the final admittance becomes $\bar{Y}'_{in} = 1$, and thus, $Z_{in} = 50\ (\Omega)$.

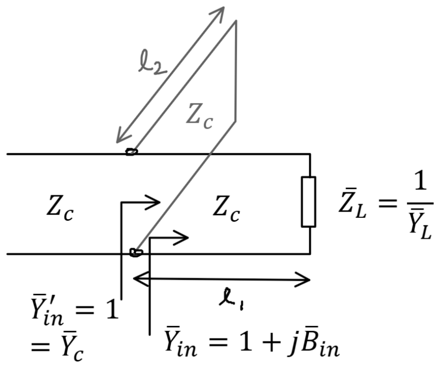

Figure 3.18: Matching with a single stub and a variable length line, all having the same Z_c.

3.3.1 Example

Given $Z_L = 150 - j100$ (Ω), match this load to a $Z_c = 50$ (Ω) transmission line at $f = 1$ (GHz), using a transmission line of length ℓ_1, and a stub of length ℓ_2, all having the characteristics impedance Z_c (assume all air-filled transmission lines).

Solution

We begin by locating the normalized admittance $\bar{Y}_L$ on the Smith chart. To do this, we first compute the normalized impedance:

$$\bar{Z}_L = \frac{Z_L}{Z_c} = \frac{150 - j100}{50} = 3 - j2 \; (\Omega) \tag{3.29}$$

The point corresponding to $\bar{Y}_L$ lies diametrically opposite $\bar{Z}_L$ on the same VSWR circle. This position is illustrated in Fig.3.19.

We can either read the value directly from the chart or calculate it:

$$\bar{Y}_L = \frac{1}{\bar{Z}_L} = \frac{1}{3 - j2} \approx 0.23 + j0.15 \; (\mathrm{S}) \tag{3.30}$$

From this point, we rotate along the constant VSWR circle toward the generator until intersecting the $\bar{Y} = 1$ or $\bar{G} = 1$ circle. At this position, the line length ℓ_1 is:

$$\frac{\ell_1}{\lambda} = 0.18 - 0.025 = 0.155. \tag{3.31}$$

Thus, the physical length is:

$$\ell_1 = 0.155 \times 30cm = 4.65 \; (\mathrm{cm}). \tag{3.32}$$

At this new point, the normalized input admittance becomes: $\bar{Y}_{in} = 1 + j1.64$ (S).

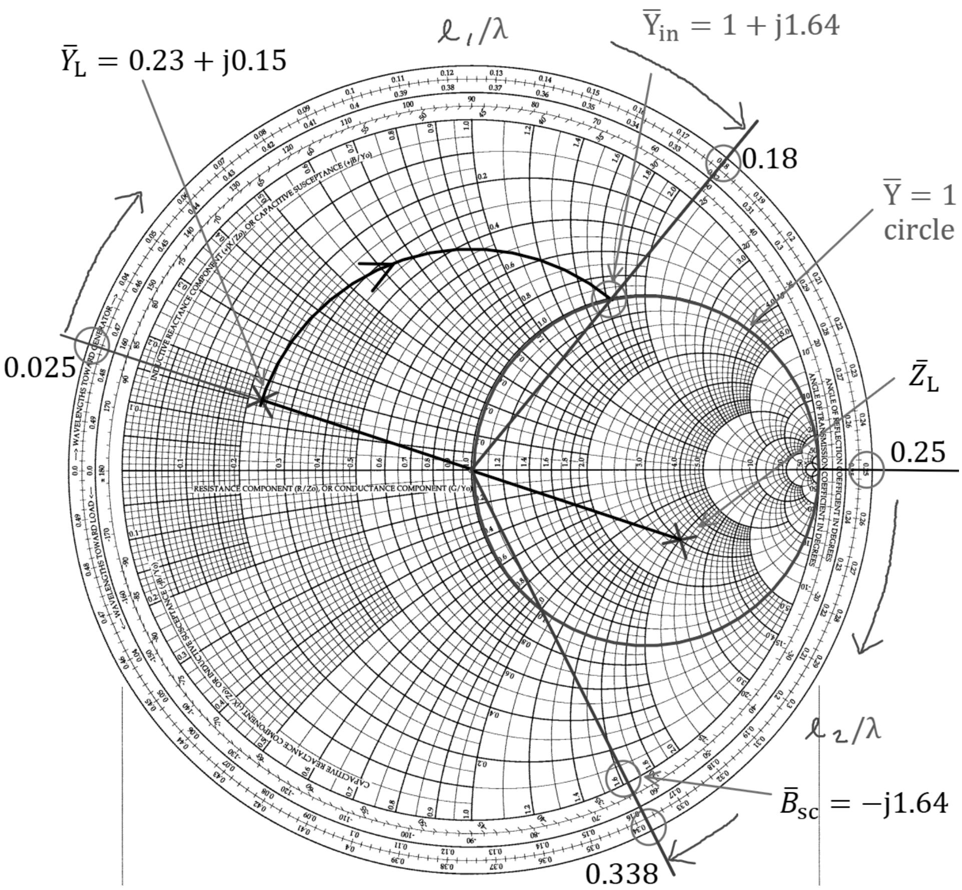

Figure 3.19: Smith chart solution of the matching problem using a single stub and a variable length line, all having the same Z_c.

To cancel the reactive part $j1.64$, we attach a short-circuited stub in parallel that supplies a susceptance of $\bar{B}_{sc} = -j1.64$. Using the Smith chart admittance view, we start at the short-circuit location (rightmost point) and rotate clockwise along a constant-conductance circle until reaching $\bar{B}_{sc} = -j1.64$. From this, the stub length is:

$$\ell_2 = (0.338 - 0.25) \times \lambda = 0.088 \times 30 = 2.64 \text{ (cm)}. \tag{3.33}$$

ADS Simulations

We first convert the stub lengths to electrical degrees:
For ℓ_1:

$$\ell_1 = 4.65 \times \frac{360}{30} = 55.8^{\circ}. \tag{3.34}$$

For ℓ_2:

$$\ell_2 = 2.64 \times \frac{360}{30} = 31.68^{\circ}. \tag{3.35}$$

The corresponding ADS schematic is shown in Fig.3.20, using the parameters calculated in the analytical solution.

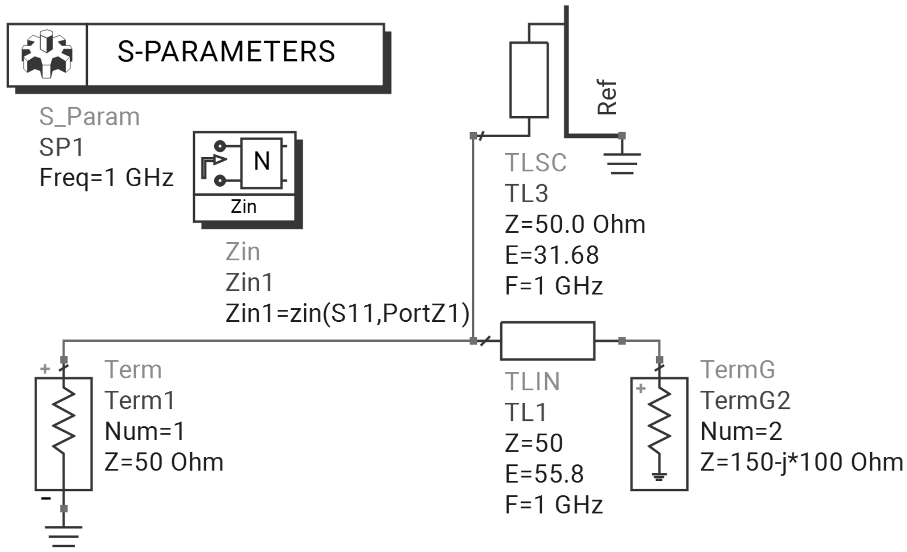

Figure 3.20: ADS circuit schematics for the short-circuited stub matching problem using lines with all Z_c.

Simulation results, presented in Fig.3.21, show excellent agreement with the desired characteristic impedance $Z_c = 50$ (Ω), confirming successful matching.

freq	Zin1	mag(S(1,1))
1.000 GHz	48.816 - j1.495	0.019

Figure 3.21: ADS results for the short-circuited stub matching problem using lines with all Z_c.

An alternative matching method using only transmission lines with characteristic impedance Z_c, known as double-stub matching, is also available. However, to maintain clarity and focus, this technique is not covered here.

3.4 LC-Matching

Electronic systems (or circuits) are typically cascaded as shown in Fig.3.22. To ensure maximum power transfer between each circuit stage, impedance matching is required — otherwise, energy is wasted across successive blocks. The condition for maximum power transfer is:

$$\boxed{Z_0 = Z_0^*}. \tag{3.36}$$

Generally, electronic systems (or circuits) are cascaded as shown in Fig.3.22. When the impedances are purely real (i.e., resistive), this sim-

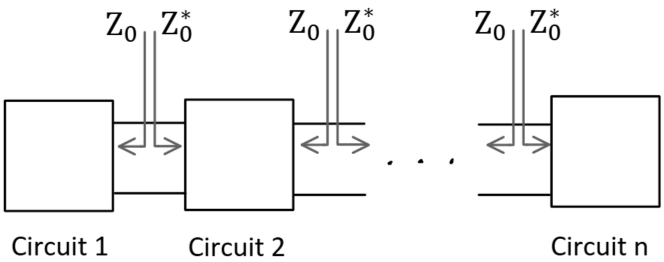

Figure 3.22: Matching for cascaded systems.

plifies to:

$$\boxed{R_0 = R_0}. \tag{3.37}$$

In practical scenarios, however, we usually encounter mismatched impedances — either $R \neq R_0$ or $Z \neq Z_0^*$. In such cases, we employ an LC matching circuit to satisfy the condition $Z_0 = Z_0^*$.

This approach involves using a pair of lumped reactive components — a capacitor and an inductor — connected in either a series or parallel configuration. Because lumped-element components are used, this technique is most effective at lower RF frequencies. For higher microwave frequencies, the transmission line matching methods introduced earlier are more suitable.

In LC matching, we aim to match two real impedances, R_{11} and R_{12}, as illustrated in Fig.3.23. In this configuration, $\boxed{R_{11} > R_{12}}$, and the components X_2 and X_3 represent reactances. The element X_2 may be either a capacitor or an inductor, but X_3 must be its dual (i.e., an inductor if X_2 is a capacitor, and vice versa).

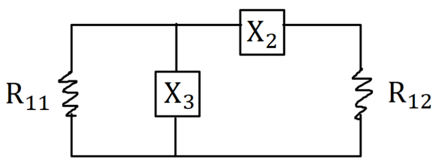

Figure 3.23: LC-matching topology.

The required values of X_2 and X_3 can be calculated using:

$$\boxed{X_2 = \mp\sqrt{R_{12}(R_{11} - R_{12})}}, \quad \boxed{X_3 = \pm R_{11}\sqrt{\frac{R_{12}}{R_{11} - R_{12}}}}. \tag{3.38}$$

3.4.1 Example

Match a 200 (Ω) load to a 50 (Ω) transmission line using an LC matching circuit. The operating frequency is 144 (MHz).

Solution

In this case, we have $R_{11} = 200$ (Ω), and $R_{12} = 50$ (Ω). Using the equations from (3.38), we first calculate:

$$\begin{aligned} X_2 &= -\sqrt{R_{12}(R_{11} - R_{12})} \\ &= -\sqrt{50(200 - 50)} \\ &= -86.6\ (\Omega), \end{aligned} \tag{3.39}$$

and

$$\begin{aligned} X_3 &= R_{11}\sqrt{\frac{R_{12}}{R_{11} - R_{12}}} \\ &= 200\sqrt{\frac{50}{200 - 50}} \\ &= 115.47\ (\Omega). \end{aligned} \tag{3.40}$$

Since X_2 is realized using a capacitor, we can write:

$$X_2 = 86.6 = \frac{1}{\omega C}, \tag{3.41}$$

where $\omega = 2\pi f$. Substituting $f = 144$ (MHz), we find C=12.7 (pF) $\approx$ 15 (pF).

Similarly, for X_3, if implemented as an inductor:

$$X_3 = 115.47 = \omega L, \tag{3.42}$$

from which we obtain $L = 127$ (nH) ≈ 120 (nH).

ADS Simulations

The ADS circuit schematics is shown in Fig.3.24. The results are shown in Fig.3.25, where the reflections at the input are minimum at $f = 144$ (MHz), as expected.

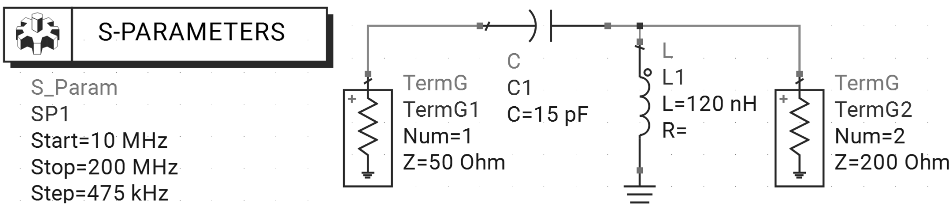

Figure 3.24: LC-matching circuit schematics.

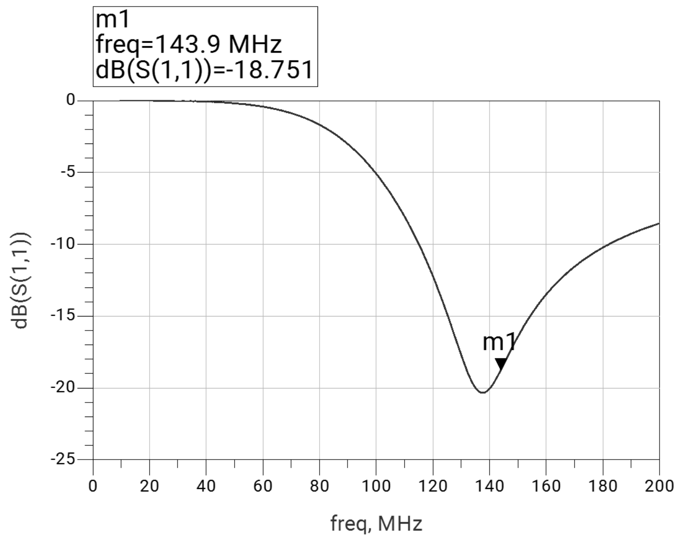

Figure 3.25: LC-matching circuit ADS simulation results.

3.5 Coil Transformer Matching

Transformers composed of wound coils can also be used for impedance matching. This technique is particularly effective at relatively low RF frequencies, such as in the HF band. At higher frequencies, increased coupling to surrounding structures makes coil transformer use less practical.

Consider the circuit shown in Fig.3.26, where an ideal transformer is placed between the source and the load impedance.

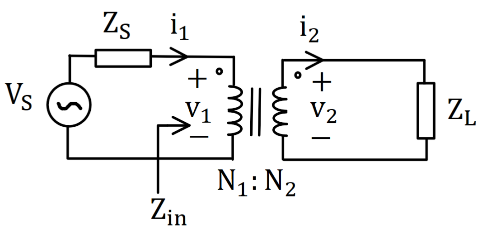

Figure 3.26: Transformer matching circuit topology.

The voltage and current relations for an ideal transformer are:

$$v_1 = \frac{v_2}{a}, \tag{3.43}$$

and

$$i_1 = a i_2, \tag{3.44}$$

where

$$\boxed{a = \frac{N_2}{N_1}}, \tag{3.45}$$

and N_1 and N_2 are the number of turns on the primary and secondary windings, respectively.

The impedance seen by the source is given by:

$$Z_{in} = \frac{v_1}{i_1}. \tag{3.46}$$

Substituting (3.43) and (3.44) into (3.46) yields:

$$Z_{in} = \frac{1}{a^2}\frac{v_2}{i_2}. \tag{3.47}$$

Recognizing that $v_2/i_2 = Z_L$, the load impedance, we find:

$$\boxed{Z_{in} = \frac{1}{a^2} Z_L}. \tag{3.48}$$

This final expression allows us to calculate the necessary turns ratio (a) for impedance matching using a transformer.

3.5.1 Example

Match a 200 (Ω) load (antenna) to a 50 (Ω) transmission line using a transformer matching circuit. The operating frequency is in the HF band.

Solution

From Equation (3.48):

$$Z_{in} = \frac{1}{a^2} Z_L. \tag{3.49}$$

Given $Z_L = 200$ (Ω) and the required $Z_{in} = 50$ (Ω) for impedance matching, we solve for (a):

$$a = \sqrt{\frac{200}{50}} = 2. \tag{3.50}$$

This implies a turns ratio of $N_1/N_2 = 0.5$, which can be realized, for example, using $N_1 = 5$ turns and $N_2 = 10$ turns. Using very low turn counts, such as $N_1 = 2$ and $N_2 = 1$, may result in practical inaccuracies due to limitations in winding precision and transformer efficiency.

ADS Simulations

The ADS circuit schematic is shown in Fig.3.27. In the simulation, the transformer turns ratio is defined as $T = N_1/N_2$ and parameterized as "nvar". The simulation results, presented in Fig.3.28, confirm that optimal impedance matching is achieved at $T = 0.5$, consistent with theoretical results.

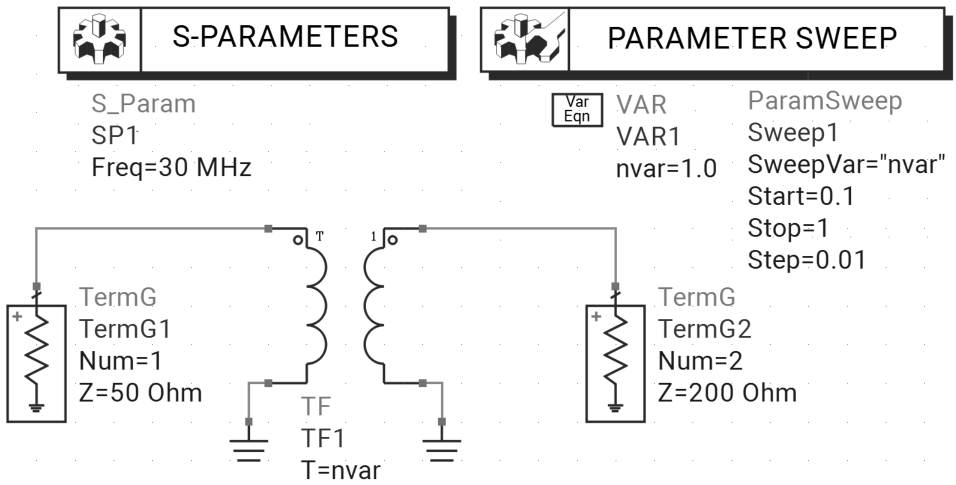

Figure 3.27: Transformer matching circuit topology.

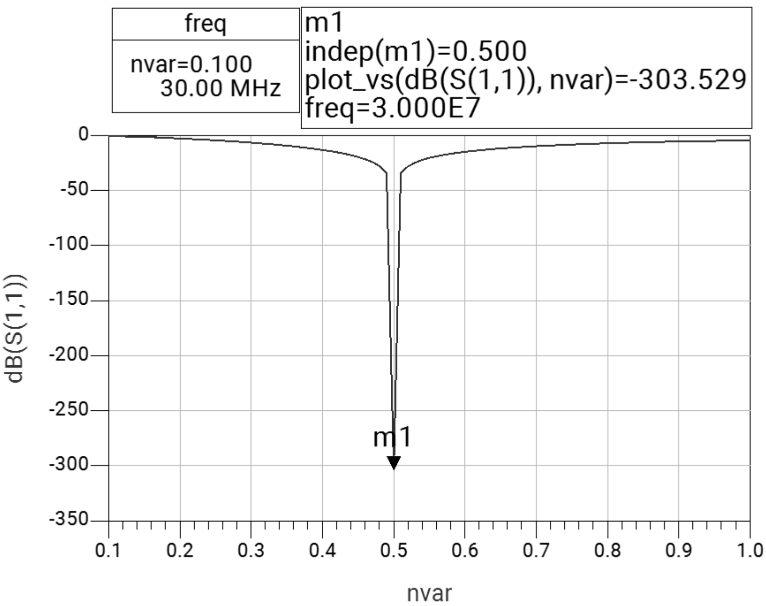

Figure 3.28: Transformer matching circuit topology.

Fig.3.29 displays a photo of a 1:2 HF toroidal antenna transformer corresponding to this configuration. This transformer is designed for use with an end-fed half-wave (EFHW) antenna. A coaxial cable connects on one side, while the antenna's dipole arms are attached to the connectors on the opposite end. The total antenna length is tuned to be half the wavelength of the system's lowest operating frequency.

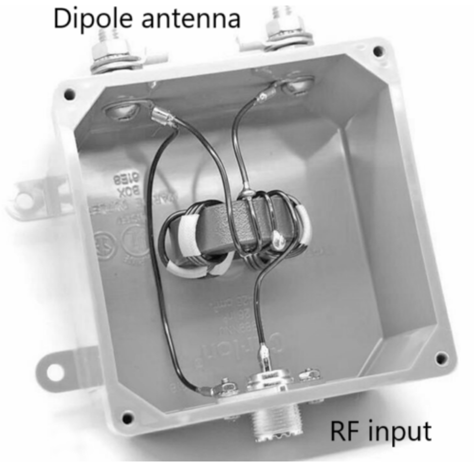

Figure 3.29: 1:2 HF antenna transformer (BALUN).

3.6 Problems

1. Match the load impedance $Z_L = 25 - j50$ (Ω) to a transmission line with characteristic impedance $Z_c = 50$ (Ω) using a series transmission line section of length ℓ_1, followed by a quarter-wave ($\lambda/4$) transformer. Assume the operating frequency is f=1 (GHz). Show your solution on the Smith chart.

2. Match the load impedance $Z_L = 25 - j50\,\Omega$ to a transmission line with characteristic impedance $Z_c = 50\,\Omega$ using a shunt stub of length ℓ_2, followed by a quarter-wave ($\lambda/4$) transformer. Assume the operating frequency is f=1 (GHz). Show your solution on the Smith chart.

3. Match the load impedance $Z_L = 25 - j50\ \Omega$ to a transmission line with characteristic impedance $Z_c = 50\ \Omega$ using an LC matching network. Assume an operating frequency of f=1 (GHz).

4. - Match the load impedance $Z_L = 25 - j50\ \Omega$ to a transmission line with characteristic impedance $Z_c = 50\ \Omega$ using a coil transformer. Assume the operating frequency is f=1 (GHz).